Name:-------------------------------

SL MATH YEAR 1 EXPONENTS DATE: ________________

Syllabus reference: 1.2,

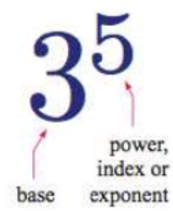
A EXPONENTS

Rather than writing $3 \times 3 \times 3 \times 3 \times 3$, we can write this product as 3^5.

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \ldots\ldots \times a}_{n \text{ factors}}$$

We say that a is the **base**, and n is the **exponent** or **index**.



NEGATIVE BASES

$(-1)^1 = -1$
$(-1)^2 = -1 \times -1 = 1$
$(-1)^3 = -1 \times -1 \times -1 = -1$
$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$

$(-2)^1 = -2$
$(-2)^2 = -2 \times -2 = 4$
$(-2)^3 = -2 \times -2 \times -2 = -8$
$(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

From the patterns above we can see that:

A **negative** base raised to an **odd** exponent is **negative**.
A **negative** base raised to an **even** exponent is **positive**.

HISTORICAL NOTE

Nicomachus discovered an interesting number pattern involving cubes and sums of odd numbers. Nicomachus was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek and was a Pythagorean.

$1 = 1^3$
$3 + 5 = 8 = 2^3$
$7 + 9 + 11 = 27 = 3^3$
$\vdots$

B LAWS OF EXPONENTS

The **exponent laws** for $m, n \in \mathbb{Z}$ are:

$a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the exponents.
$\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	To **divide** numbers with the same base, keep the base and **subtract** the exponents.
$(a^m)^n = a^{m \times n}$	When **raising** a **power** to a **power**, keep the base and **multiply** the exponents.
$(ab)^n = a^n b^n$	The power of a product is the product of the powers.
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$	The power of a quotient is the quotient of the powers.
$a^0 = 1, \quad a \neq 0$	Any non-zero number raised to the power of zero is 1.

$a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$ and in particular $a^{-1} = \dfrac{1}{a}, \quad a \neq 0.$

Example 3

Simplify using the exponent laws: **a** $3^5 \times 3^4$ **b** $\dfrac{5^3}{5^5}$ **c** $\left(m^4\right)^3$

Example 4

Write as powers of 2:

a 16 **b** $\frac{1}{16}$ **c** 1 **d** 4×2^n **e** $\frac{2^m}{8}$

EXERCISE 3B

1 Simplify using the laws of exponents:

a $5^4 \times 5^7$ **b** $d^2 \times d^6$ **c** $\frac{k^8}{k^3}$ **d** $\frac{7^5}{7^6}$ **e** $\left(x^2\right)^5$ **f** $\left(3^4\right)^4$

g $\frac{p^3}{p^7}$ **h** $n^3 \times n^9$ **i** $\left(5^t\right)^3$ **j** $7^x \times 7^2$ **k** $\frac{10^3}{10^q}$ **l** $\left(c^4\right)^m$

2 Write as powers of 2:

a 4 **b** $\frac{1}{4}$ **c** 8 **d** $\frac{1}{8}$ **e** 32 **f** $\frac{1}{32}$

g 2 **h** $\frac{1}{2}$ **i** 64 **j** $\frac{1}{64}$ **k** 128 **l** $\frac{1}{128}$

3 Write as powers of 3:

a 9 **b** $\frac{1}{9}$ **c** 27 **d** $\frac{1}{27}$ **e** 3 **f** $\frac{1}{3}$

g 81 **h** $\frac{1}{81}$ **i** 1 **j** 243 **k** $\frac{1}{243}$

4 Write as a single power of 2:

a 2×2^a **b** 4×2^b **c** 8×2^t **d** $(2^{x+1})^2$ **e** $(2^{1-n})^{-1}$

f $\frac{2^c}{4}$ **g** $\frac{2^m}{2^{-m}}$ **h** $\frac{4}{2^{1-n}}$ **i** $\frac{2^{x+1}}{2^x}$ **j** $\frac{4^x}{2^{1-x}}$

5 Write as a single power of 3:

a 9×3^p **b** 27^a **c** 3×9^n **d** 27×3^d **e** 9×27^t

f $\frac{3^y}{3}$ **g** $\frac{3}{3^y}$ **h** $\frac{9}{27^t}$ **i** $\frac{9^a}{3^{1-a}}$ **j** $\frac{9^{n+1}}{3^{2n-1}}$

.

Example 5

Write in simplest form, without brackets:

a $\left(-3a^2\right)^4$

b $\left(-\frac{2a^2}{b}\right)^3$

6 Write without brackets:

a $(2a)^2$

b $(3b)^3$

c $(ab)^4$

d $(pq)^3$

e $\left(\frac{m}{n}\right)^2$

f $\left(\frac{a}{3}\right)^3$

g $\left(\frac{b}{c}\right)^4$

h $\left(\frac{2a}{b}\right)^0$

i $\left(\frac{m}{3n}\right)^4$

j $\left(\frac{xy}{2}\right)^3$

7 Write the following in simplest form, without brackets:

a $(-2a)^2$

b $(-6b^2)^2$

c $(-2a)^3$

d $(-3m^2n^2)^3$

e $(-2ab^4)^4$

f $\left(\frac{-2a^2}{b^2}\right)^3$

g $\left(\frac{-4a^3}{b}\right)^2$

h $\left(\frac{-3p^2}{q^3}\right)^2$

Example 6

Write without negative exponents: $\dfrac{a^{-3}b^{2}}{c^{-1}}$

8 Write without negative exponents:

a ab^{-2} **b** $(ab)^{-2}$ **c** $(2ab^{-1})^{2}$ **d** $(3a^{-2}b)^{2}$ **e** $\dfrac{a^{2}b^{-1}}{c^{2}}$

f $\dfrac{a^{2}b^{-1}}{c^{-2}}$ **g** $\dfrac{1}{a^{-3}}$ **h** $\dfrac{a^{-2}}{b^{-3}}$ **i** $\dfrac{2a^{-1}}{d^{2}}$ **j** $\dfrac{12a}{m^{-3}}$

Example 7

Write $\dfrac{1}{2^{1-n}}$ in non-fractional form.

9 Write in non-fractional form:

a $\dfrac{1}{a^n}$ b $\dfrac{1}{b^{-n}}$ c $\dfrac{1}{3^{2-n}}$ d $\dfrac{a^n}{b^{-m}}$ e $\dfrac{a^{-n}}{a^{2+n}}$

10 Simplify, giving your answers in simplest rational form:

a $\left(\frac{5}{3}\right)^0$ b $\left(\frac{7}{4}\right)^{-1}$ c $\left(\frac{1}{6}\right)^{-1}$ d $\dfrac{3^3}{3^0}$

e $\left(\frac{4}{3}\right)^{-2}$ f $2^1 + 2^{-1}$ g $\left(1\frac{2}{3}\right)^{-3}$ h $5^2 + 5^1 + 5^{-1}$

11 Write as powers of 2, 3 and/or 5:

a $\frac{1}{9}$ b $\frac{1}{16}$ c $\frac{1}{125}$ d $\frac{3}{5}$

e $\frac{4}{27}$ f $\dfrac{2^c}{8 \times 9}$ g $\dfrac{9^k}{10}$ h $\dfrac{6^p}{75}$

12 Read about Nicomachus' pattern on page **84** and find the series of odd numbers for:

a 5^3 b 7^3 c 12^3

EXERCISE 3B

1 a 5^{11} b d^8 c k^5 d $\frac{1}{7}$ e x^{10} f 3^{16}
g p^{-4} h n^{12} i 5^{3t} j 7^{x+2} k 10^{3-q} l c^{4m}

2 a 2^2 b 2^{-2} c 2^3 d 2^{-3} e 2^5 f 2^{-5}
g 2^1 h 2^{-1} i 2^6 j 2^{-6} k 2^7 l 2^{-7}

3 a 3^2 b 3^{-2} c 3^3 d 3^{-3} e 3^1 f 3^{-1}
g 3^4 h 3^{-4} i 3^0 j 3^5 k 3^{-5}

4 a 2^{a+1} b 2^{b+2} c 2^{t+3} d 2^{2x+2} e 2^{n-1}
f 2^{c-2} g 2^{2m} h 2^{n+1} i 2^1 j 2^{3x-1}

5 a 3^{p+2} b 3^{3a} c 3^{2n+1} d 3^{d+3} e 3^{3t+2}
f 3^{y-1} g 3^{1-y} h 3^{2-3t} i 3^{3a-1} j 3^3

6 a $4a^2$ b $27b^3$ c a^4b^4 d p^3q^3 e $\frac{m^2}{n^2}$
f $\frac{a^3}{27}$ g $\frac{b^4}{c^4}$ h $1, b \neq 0$ i $\frac{m^4}{81n^4}$ j $\frac{x^3y^3}{8}$

7 a $4a^2$ b $36b^4$ c $-8a^3$ d $-27m^6n^6$
e $16a^4b^{16}$ f $\frac{-8a^6}{b^6}$ g $\frac{16a^6}{b^2}$ h $\frac{9p^4}{q^6}$

8 a $\frac{a}{b^2}$ b $\frac{1}{a^2b^2}$ c $\frac{4a^2}{b^2}$ d $\frac{9b^2}{a^4}$ e $\frac{a^2}{bc^2}$
f $\frac{a^2c^2}{b}$ g a^3 h $\frac{b^3}{a^2}$ i $\frac{2}{ad^2}$ j $12am^3$

9 a a^{-n} b b^n c 3^{n-2} d a^nb^m e a^{-2n-2}

10 a 1 b $\frac{4}{7}$ c 6 d 27 e $\frac{9}{16}$ f $\frac{5}{2}$
g $\frac{27}{125}$ h $\frac{151}{5}$

11 a 3^{-2} b 2^{-4} c 5^{-3} d $3^1 \times 5^{-1}$ e $2^2 \times 3^{-3}$
f $2^{c-3} \times 3^{-2}$ g $3^{2k} \times 2^{-1} \times 5^{-1}$ h $2^p \times 3^{p-1} \times 5^{-2}$

12 a $5^3 = 21 + 23 + 25 + 27 + 29$
b $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$
c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

C RATIONAL EXPONENTS

The exponent laws used previously can also be applied to **rational exponents**, or exponents which are written as a fraction.

For $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {exponent laws}
and $\sqrt{a} \times \sqrt{a} = a$ also.

So, $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ {by direct comparison}

Likewise $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$
and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$

In general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads 'the nth root of a', for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m}$
$= (a^m)^{\frac{1}{n}}$
$= a^{\frac{m}{n}}$

$\therefore$ $\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$ for $a > 0$, $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$

Example 8

Write as a single power of 2: **a** $\sqrt[3]{2}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

EXERCISE 3C

1 Write as a single power of 2:

a $\sqrt[5]{2}$ b $\frac{1}{\sqrt[5]{2}}$ c $2\sqrt{2}$ d $4\sqrt{2}$ e $\frac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ g $\frac{4}{\sqrt{2}}$ h $(\sqrt{2})^3$ i $\frac{1}{\sqrt[3]{16}}$ j $\frac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ b $\frac{1}{\sqrt[3]{3}}$ c $\sqrt[4]{3}$ d $3\sqrt{3}$ e $\frac{1}{9\sqrt{3}}$

Example 9

Use your calculator to evaluate:

a $2^{\frac{7}{5}}$ **b** $\frac{1}{\sqrt[3]{4}}$

Example 10

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

3 Write the following in the form a^x where a is a prime number and x is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\frac{1}{\sqrt[3]{7}}$ **g** $\frac{1}{\sqrt[4]{27}}$ **h** $\frac{1}{\sqrt[5]{16}}$ **i** $\frac{1}{\sqrt[3]{32}}$ **j** $\frac{1}{\sqrt[7]{49}}$

4 Use your calculator to find:

a $3^{\frac{3}{4}}$ **b** $2^{\frac{7}{8}}$ **c** $2^{-\frac{1}{3}}$ **d** $4^{-\frac{3}{5}}$ **e** $\sqrt[4]{8}$

f $\sqrt[5]{27}$ **g** $\frac{1}{\sqrt[3]{7}}$

5 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

EXERCISE 3C

1	**a** $2^{\frac{1}{5}}$	**b** $2^{-\frac{1}{5}}$	**c** $2^{\frac{3}{2}}$	**d** $2^{\frac{5}{2}}$	**e** $2^{-\frac{1}{3}}$
	f $2^{\frac{4}{3}}$	**g** $2^{\frac{3}{2}}$	**h** $2^{\frac{3}{2}}$	**i** $2^{-\frac{4}{3}}$	**j** $2^{-\frac{3}{2}}$
2	**a** $3^{\frac{1}{3}}$	**b** $3^{-\frac{1}{3}}$	**c** $3^{\frac{1}{4}}$	**d** $3^{\frac{3}{2}}$	**e** $3^{-\frac{5}{2}}$
3	**a** $7^{\frac{1}{3}}$	**b** $3^{\frac{3}{4}}$	**c** $2^{\frac{4}{5}}$	**d** $2^{\frac{5}{3}}$	**e** $7^{\frac{2}{7}}$
	f $7^{-\frac{1}{3}}$	**g** $3^{-\frac{3}{4}}$	**h** $2^{-\frac{4}{5}}$	**i** $2^{-\frac{5}{3}}$	**j** $7^{-\frac{2}{7}}$
4	**a** 2.28	**b** 1.83	**c** 0.794	**d** 0.435	**e** 1.68
	f 1.93	**g** 0.523			
5	**a** 8	**b** 32	**c** 8	**d** 125	**e** 4
	f $\frac{1}{2}$	**g** $\frac{1}{27}$	**h** $\frac{1}{16}$	**i** $\frac{1}{81}$	**j** $\frac{1}{25}$

D ALGEBRAIC EXPANSION AND FACTORISATION

EXPANSION

We can use the usual expansion laws to simplify expressions containing exponents:

$$\begin{aligned} a(b+c) &= ab+ac \\ (a+b)(c+d) &= ac+ad+bc+bd \\ (a+b)(a-b) &= a^2-b^2 \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a-b)^2 &= a^2-2ab+b^2 \end{aligned}$$

Example 11

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

EXERCISE 3D.1

1 Expand and simplify:

- **a** $x^2(x^3+2x^2+1)$
- **b** $2^x(2^x+1)$
- **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$
- **d** $7^x(7^x+2)$
- **e** $3^x(2-3^{-x})$
- **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$
- **g** $2^{-x}(2^x+5)$
- **h** $5^{-x}(5^{2x}+5^x)$
- **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$

Example 12

Expand and simplify: **a** $(2^x+3)(2^x+1)$ **b** $(7^x+7^{-x})^2$

2 Expand and simplify:

a $(2^x-1)(2^x+3)$

b $(3^x+2)(3^x+5)$

c $(5^x-2)(5^x-4)$

d $(2^x+3)^2$

e $(3^x-1)^2$

f $(4^x+7)^2$

g $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$

h $(2^x+3)(2^x-3)$

i $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$

j $(x+\frac{2}{x})^2$

k $(7^x-7^{-x})^2$

l $(5-2^{-x})^2$

EXERCISE 3D.1

1. **a** $x^5+2x^4+x^2$ **b** 4^x+2^x **c** $x+1$
   d $49^x+2(7^x)$ **e** $2(3^x)-1$ **f** x^2+2x+3
   g $1+5(2^{-x})$ **h** 5^x+1 **i** $x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$
2. **a** $4^x+2^{x+1}-3$ **b** $9^x+7(3^x)+10$
   c $25^x-6(5^x)+8$ **d** $4^x+6(2^x)+9$
   e $9^x-2(3^x)+1$ **f** $16^x+14(4^x)+49$
   g $x-4$ **h** 4^x-9 **i** $x-x^{-1}$ **j** $x^2+4+\dfrac{4}{x^2}$
   k $7^{2x}-2+7^{-2x}$ **l** $25-10(2^{-x})+4^{-x}$

FACTORISATION AND SIMPLIFICATION

Example 13

Factorise: **a** $2^{n+3}+2^n$ **b** $2^{n+3}+8$ **c** $2^{3n}+2^{2n}$

EXERCISE 3D.2

1. Factorise:
   a $5^{2x}+5^x$ **b** $3^{n+2}+3^n$ **c** 7^n+7^{3n}
   d $5^{n+1}-5$ **e** $6^{n+2}-6$ **f** $4^{n+2}-16$

Example 14

Factorise: **a** $4^x - 9$ **b** $9^x + 4(3^x) + 4$

2 Factorise:

a $9^x - 4$
b $4^x - 25$
c $16 - 9^x$
d $25 - 4^x$
e $9^x - 4^x$
f $4^x + 6(2^x) + 9$
g $9^x + 10(3^x) + 25$
h $4^x - 14(2^x) + 49$
i $25^x - 4(5^x) + 4$

3 Factorise:

a $4^x + 9(2^x) + 18$
b $4^x - 2^x - 20$
c $9^x + 9(3^x) + 14$
d $9^x + 4(3^x) - 5$
e $25^x + 5^x - 2$
f $49^x - 7^{x+1} + 12$

Example 15

Simplify: a $\frac{6^n}{3^n}$ b $\frac{4^n}{6^n}$

4 Simplify:

a $\frac{12^n}{6^n}$ b $\frac{20^a}{2^a}$ c $\frac{6^b}{2^b}$ d $\frac{4^n}{20^n}$

e $\frac{35^x}{7^x}$ f $\frac{6^a}{8^a}$ g $\frac{5^{n+1}}{5^n}$ h $\frac{5^{n+1}}{5}$

Example 16

Simplify: a $\frac{3^n + 6^n}{3^n}$ b $\frac{2^{m+2} - 2^m}{2^m}$ c $\frac{2^{m+3} + 2^m}{9}$

5 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$

b $\dfrac{2^n + 12^n}{2^n}$

c $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$

e $\dfrac{6^n + 12^n}{1 + 2^n}$

f $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$

h $\dfrac{4^n - 2^n}{2^n}$

i $\dfrac{2^n - 2^{n-1}}{2^n}$

6 Simplify:

a $2^n(n+1)+2^n(n-1)$

b $3^n\left(\frac{n-1}{6}\right)-3^n\left(\frac{n+1}{6}\right)$

EXERCISE 3D.2

1 **a** $5^x(5^x+1)$ **b** $10(3^n)$ **c** $7^n(1+7^{2n})$
d $5(5^n-1)$ **e** $6(6^{n+1}-1)$ **f** $16(4^n-1)$

2 **a** $(3^x+2)(3^x-2)$ **b** $(2^x+5)(2^x-5)$ **c** $(4+3^x)(4-3^x)$
d $(5+2^x)(5-2^x)$ **e** $(3^x+2^x)(3^x-2^x)$ **f** $(2^x+3)^2$
g $(3^x+5)^2$ **h** $(2^x-7)^2$ **i** $(5^x-2)^2$

3 **a** $(2^x+3)(2^x+6)$ **b** $(2^x+4)(2^x-5)$
c $(3^x+2)(3^x+7)$ **d** $(3^x+5)(3^x-1)$
e $(5^x+2)(5^x-1)$ **f** $(7^x-4)(7^x-3)$

4 **a** 2^n **b** 10^a **c** 3^b **d** $\frac{1}{5^n}$ **e** 5^x
f $(\frac{3}{4})^a$ **g** 5 **h** 5^n

5 **a** 3^m+1 **b** $1+6^n$ **c** 4^n+2^n **d** 4^x-1
e 6^n **f** 5^n **g** 4 **h** 2^n-1 **i** $\frac{1}{2}$

6 **a** $n\,2^{n+1}$ **b** -3^{n-1}