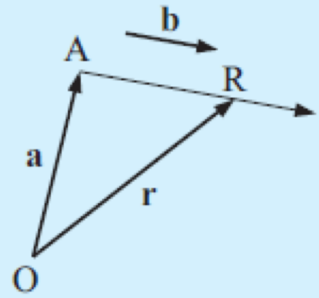
D CONSTANT VELOCITY PROBLEMS

If an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\mathbf{b}|$.



$$\vec{V} = \frac{\Delta \vec{d}}{t} = \frac{\vec{d}_f - \vec{d}_i}{t}$$

$$\vec{v}\,t = \vec{d}_f - \vec{d}_i$$

$$\vec{d}_i + \vec{v}\,t = \vec{d}_f$$

$$\vec{a} + \vec{B}\,t = \vec{r}$$

d = distance

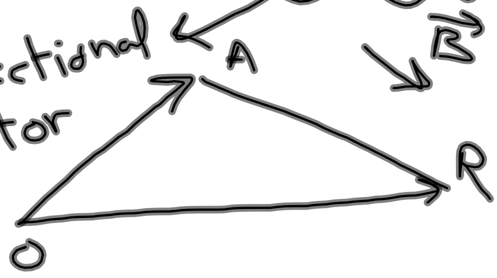
v = velocity

t = time

d_i → initial
f → final

$$\vec{r} = \vec{a} + t\vec{B}$$

directional vector



$\vec{r} = \vec{a} + t\vec{\beta} \Leftrightarrow \vec{d_f} = \vec{d_i} + \vec{v}t$

Example 8 **Self Tutor**

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the vector equation of the path of an object.

The time t is in seconds, $t \geqslant 0$. The distance units are metres.

a Find the object's initial position. $t = 0 \Rightarrow d_i = (1, 9)$

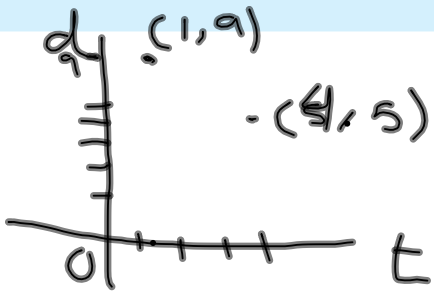
b Plot the path of the object for $t = 0, 1, 2, 3$.

c Find the velocity vector of the object.

d Find the object's speed.

e If the object continues in the same direction but increases its speed to 30 m s^{-1}, state its new velocity vector.

$\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$



$\text{Speed} = |\vec{v}| = \sqrt{3^2 + (-4)^2} = 5$

$\boxed{e}$ Speed initial $\longrightarrow$ Speed final

$5 \text{ m/s} \xrightarrow{\boxed{\times 6}} 30 \text{ m/s}$

Velocity initial $\longrightarrow$ Velocity final

$\begin{pmatrix} 3 \\ -4 \end{pmatrix} \longrightarrow 6\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$

Example 9 **Self Tutor**

An object is initially at $(5, 10)$ and moves with velocity vector $3\mathbf{i} - \mathbf{j}$ metres per minute. Find:

- **a** the position of the object at time t minutes
- **b** the speed of the object
- **c** the position of the object at $t = 3$ minutes
- **d** the time when the object is due east of $(0, 0)$.

a) $\vec{r} = \vec{a} + t\vec{\beta}$

$\vec{d_f} = \vec{d_i} + t\vec{v}$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix} + t\begin{pmatrix}3\\-1\end{pmatrix}$$

b) Speed $= |\vec{V}|$

$= \sqrt{3^2 + (-1)^2}$

$= \sqrt{10}$

c) $t = 3$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix} + 3\begin{pmatrix}3\\-1\end{pmatrix}$$

$$= \begin{pmatrix}5\\10\end{pmatrix} + \begin{pmatrix}9\\-3\end{pmatrix} = \begin{pmatrix}14\\7\end{pmatrix}$$

d) $t = ?$

$d_f = \begin{pmatrix}x\\0\end{pmatrix}$

$$\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix} + t\begin{pmatrix}3\\-1\end{pmatrix}$$

$$\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix} + t\begin{pmatrix}3\\-1\end{pmatrix}$$

$0 = 10 - t$

$\Rightarrow t = 10$ min.

XERCISE 13D

1 A particle at $P(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

$\vec{p} \rightarrow \cancel{\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}2\\-5\end{pmatrix}}$

- **a** Find the initial position of P.
- **b** Illustrate the initial part of the motion of P where $t = 0, 1, 2, 3$.
- **c** Find the velocity vector of P.
- **d** Find the speed of P.

2
- **a** Find the vector equation of a boat initially at $(2, 3)$, which travels with velocity vector $\begin{pmatrix}4\\-5\end{pmatrix}$. The grid units are kilometres and the time is in hours.
- **b** Locate the boat's position after 90 minutes.
- **c** How long will it take for the boat to reach the point $(5, -0.75)$?

3 A remote controlled toy car is initially at the point $(-3, -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.

- **a** Write an expression for the position vector of the car at any time $t \geqslant 0$.
- **b** Hence find the position vector of the car at time $t = 2.5$.
- **c** Find when the car is **i** due north **ii** due west of the observation point $(0, 0)$.
- **d** Plot the car's positions at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, ...$

4 Each of the following vector equations represents the path of a moving object. t is measured in seconds and $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

- **a** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-4\\3\end{pmatrix} + t\begin{pmatrix}12\\5\end{pmatrix}$
- **b** $x = 3 + 2t$, $y = -t$, $z = 4 - 2t$

5 Find the velocity vector of a speed boat moving parallel to:

a $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with a speed of 150 km h^{-1} **b** $2\mathbf{i} + \mathbf{j}$ with a speed of 50 km h^{-1}.

6 Find the velocity vector of a swooping eagle moving in the direction $-2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}$ with a speed of 90 km h^{-1}.

7 Yacht A moves according to $x(t) = 4 + t$, $y(t) = 5 - 2t$ where the distance units are kilometres and the time units are hours. Yacht B moves according to $x(t) = 1 + 2t$, $y(t) = -8 + t$, $t \geqslant 0$.

a Find the initial position of each yacht.

b Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant, and state these speeds.

d Verify algebraically that the paths of the yachts are at right angles to each other.

8 Submarine P is at $(-5, 4)$ and fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ at 1:34 pm.

Submarine Q is at $(15, 7)$ and a minutes later fires a torpedo with velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

Distances are measured in kilometres and time is in minutes.

a Show that the position of P's torpedo can be written as $\text{P}(x_1(t), y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.

b What is the speed of P's torpedo?

c Show that the position of Q's torpedo can be written as $\text{Q}(x_2(t), y_2(t))$ where $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$.

d Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo and at what time did the explosion occur?

9 A helicopter at $\text{A}(6, 9, 3)$ moves with constant velocity in a straight line. 10 minutes later it is at $\text{B}(3, 10, 2.5)$. Distances are in kilometres.

a Find $\overrightarrow{\text{AB}}$. **b** Find the helicopter's speed.

c Determine the equation of the straight line path of the helicopter.

d The helicopter is travelling directly towards its helipad, which has z-coordinate 0. Find the total time taken for the helicopter to land.

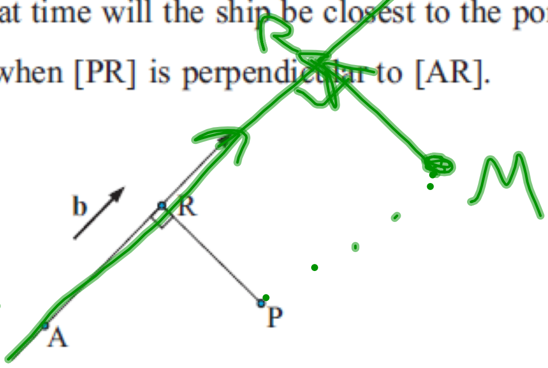
E THE SHORTEST DISTANCE FROM A LINE TO A POINT

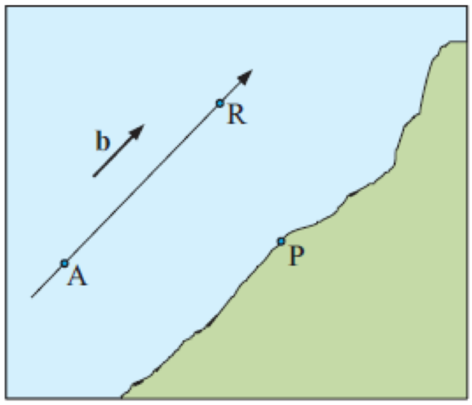
A ship R sails through point A in the direction **b** and continues past a port P. At what time will the ship be closest to the port?

The ship is closest when [PR] is perpendicular to [AR].

$\therefore \quad \overrightarrow{PR} \bullet \mathbf{b} = 0$

$\vec{r} = \vec{a} + \vec{\beta} t$



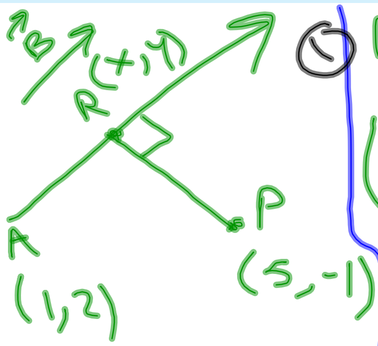


In this situation, point R is called the **foot of the perpendicular from P to the line**.

Example 10 **Self Tutor**

A line has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$. Let P be the point $(5, -1)$.

Find exactly the shortest distance from P to the line. $R - P$



① $\vec{PR} = \begin{pmatrix} x-5 \\ y+1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$x = 1 + 3t$

$y = 2 - t$

② $\vec{PR} = \begin{pmatrix} 1+3t \ -5 \\ 2-t \ +1 \end{pmatrix} = \begin{pmatrix} 3t-4 \\ 3-t \end{pmatrix}$

③ $\vec{B} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\vec{PR} = \begin{pmatrix} 3t-4 \\ 3-t \end{pmatrix}$

$\vec{B} \cdot \vec{PR}$

$\begin{pmatrix} 3 \\ -1 \end{pmatrix} \begin{pmatrix} 3t-4 \\ 3-t \end{pmatrix} = 0$

$3(3t-4) + (-1)(3-t) = 0$

$9t - 12 - 3 + t = 0$

$10t - 15 = 0$

$t = \frac{15}{10} = 1.5$

④ $\vec{PR} = \begin{pmatrix} 3(1.5)-4 \\ 3-1.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}$

⑤ $|PR|$ = Shortest

$|\vec{PR}| = \sqrt{0.5^2 + 1.5^2} = \frac{\sqrt{10}}{2}$

EXERCISE 13E

1 Find the shortest distance from:

a P(3, 2) to the line with parametric equations $x = 2 + t, \quad y = 3 + 2t, \quad t \in \mathbb{R}$

b Q(−1, 1) to the line with parametric equations $x = t, \quad y = 1 - t, \quad t \in \mathbb{R}$

c R(−3, −1) to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad s \in \mathbb{R}$

d S(5, −2) to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -7 \end{pmatrix}, \quad t \in \mathbb{R}.$