Functions - November 28, 2016 *[41 marks]*

1a. Let $f(x) = 3x - 2$ and $g(x) = \frac{5}{3x}$, for $x \neq 0$. *[2 marks]*

Find $f^{-1}(x)$.

Markscheme

interchanging x and y ***(M1)***

eg $x = 3y - 2$

$f^{-1}(x) = \frac{x+2}{3}$ $\left(\text{accept } y = \frac{x+2}{3}, \ \frac{x+2}{3}\right)$ ***A1 N2***

[2 marks]

1b. Show that $\left(g \circ f^{-1}\right)(x) = \frac{5}{x+2}$. *[2 marks]*

Markscheme

attempt to form composite (in any order) ***(M1)***

eg $g\left(\frac{x+2}{3}\right), \ \frac{\frac{5}{3x}+2}{3}$

correct substitution ***A1***

eg $\frac{5}{3\left(\frac{x+2}{3}\right)}$

$\left(g \circ f^{-1}\right)(x) = \frac{5}{x+2}$ ***AG N0***

[2 marks]

1c. Let $h(x) = \frac{5}{x+2}$, for $x \geqslant 0$. The graph of h has a horizontal asymptote at $y = 0$. *[2 marks]*

Find the y-intercept of the graph of h.

Markscheme

valid approach ***(M1)***

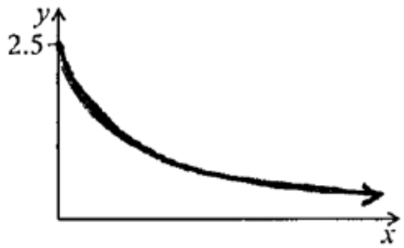
eg $h(0), \ \frac{5}{0+2}$

$y = \frac{5}{2}$ $\left(\text{accept } (0, 2.5)\right)$ ***A1 N2***

[2 marks]

1d. Hence, sketch the graph of h. *[3 marks]*

Markscheme



A1A2 *N3*

Notes: Award ***A1*** for approximately correct shape (reciprocal, decreasing, concave up).

Only if this ***A1*** is awarded, award ***A2*** for all the following approximately correct features: y-intercept at $(0, 2.5)$, asymptotic to x-axis, correct domain $x \geqslant 0$.

If only two of these features are correct, award ***A1***.

[3 marks]

1e. For the graph of h^{-1}, write down the x-intercept; *[1 mark]*

Markscheme

$x = \frac{5}{2}$ $\left(\text{accept } (2.5, 0)\right)$ ***A1*** ***N1***

[1 mark]

1f. For the graph of h^{-1}, write down the equation of the vertical asymptote. *[1 mark]*

Markscheme

$x = 0$ (must be an equation) ***A1*** ***N1***

[1 mark]

1g. Given that $h^{-1}(a) = 3$, find the value of a. *[3 marks]*

Markscheme

METHOD 1

attempt to substitute 3 into h (seen anywhere) ***(M1)***

eg $h(3),\ \frac{5}{3+2}$

correct equation ***(A1)***

eg $a = \frac{5}{3+2}\ ,\ h(3) = a$

$a = 1$ ***A1 N2***

[3 marks]

METHOD 2

attempt to find inverse (may be seen in (d)) ***(M1)***

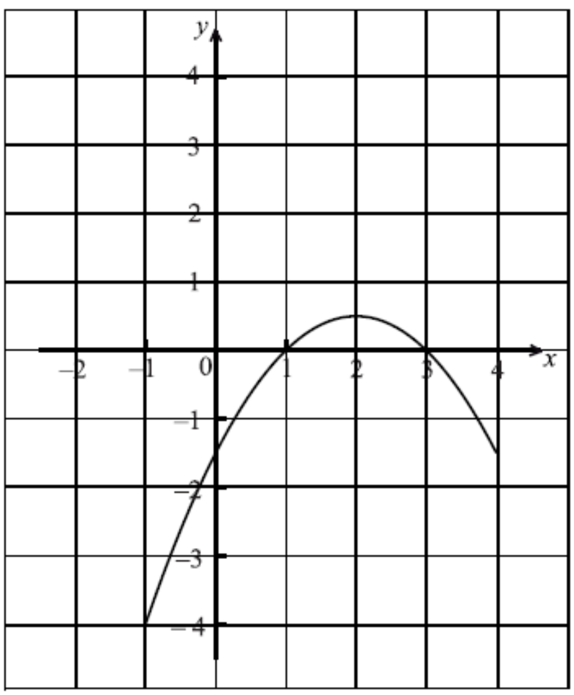
eg $x = \frac{5}{y+2}\ ,\ h^{-1} = \frac{5}{x} - 2,\ \frac{5}{x} + 2$

correct equation, $\frac{5}{x} - 2 = 3$ ***(A1)***

$a = 1$ ***A1 N2***

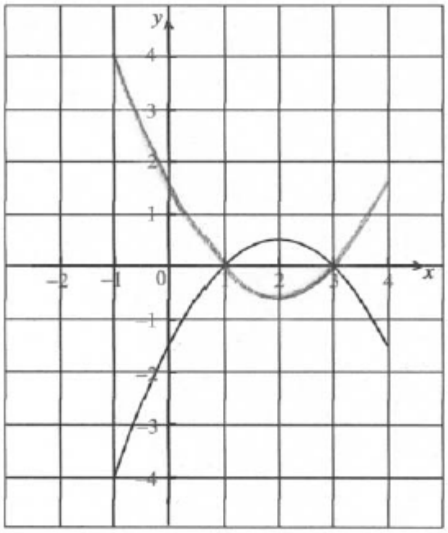
[3 marks]

2a. Part of the graph of a function f is shown in the diagram below. *[2 marks]*



On the same diagram sketch the graph of $y = -f(x)$.

Markscheme



M1A1 ***N2***

Note: Award ***M1*** for evidence of reflection in x-axis, ***A1*** for correct vertex **and** all intercepts approximately correct.

2b. Let $g(x) = f(x+3)$. *[4 marks]*

(i) Find $g(-3)$.

(ii) Describe fully the transformation that maps the graph of f to the graph of g.

Markscheme

(i) $g(-3) = f(0)$ ***(A1)***

$f(0) = -1.5$ ***A1*** ***N2***

(ii) translation (accept shift, slide, etc.) of $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ ***A1A1*** ***N2***

[4 marks]

3a. Consider $f(x) = 2kx^2 - 4kx + 1$, for $k \neq 0$. The equation $f(x) = 0$ has two equal roots. *[5 marks]*

Find the value of k .

Markscheme

valid approach ***(M1)***

e.g. $b^2 - 4ac$, $\Delta = 0$, $(-4k)^2 - 4(2k)(1)$

correct equation ***A1***

e.g. $(-4k)^2 - 4(2k)(1) = 0$, $16k^2 = 8k$, $2k^2 - k = 0$

correct manipulation ***A1***

e.g. $8k(2k - 1)$, $\frac{8 \pm \sqrt{64}}{32}$

$k = \frac{1}{2}$ ***A2*** ***N3***

[5 marks]

3b. The line $y = p$ intersects the graph of f . Find all possible values of p . *[2 marks]*

Markscheme

recognizing vertex is on the x-axis ***M1***

e.g. $(1, 0)$, sketch of parabola opening upward from the x-axis

$p \geq 0$ ***A1*** ***N1***

[2 marks]

4a. Let $f(x) = x^2$ and $g(x) = 2(x - 1)^2$. *[2 marks]*

The graph of g can be obtained from the graph of f using two transformations.

Give a full geometric description of each of the two transformations.

Markscheme

in any order

translated 1 unit to the right ***A1*** ***N1***

stretched vertically by factor 2 ***A1*** ***N1***

[2 marks]

4b. The graph of g is translated by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ to give the graph of h. *[4 marks]*

The point $(-1, 1)$ on the graph of f is translated to the point P on the graph of h.

Find the coordinates of P.

Markscheme

METHOD 1

finding coordinates of image on g ***(A1)(A1)***

e.g. $-1 + 1 = 0$, $1 \times 2 = 2$, $(-1, 1) \rightarrow (-1 + 1, 2 \times 1)$, $(0, 2)$

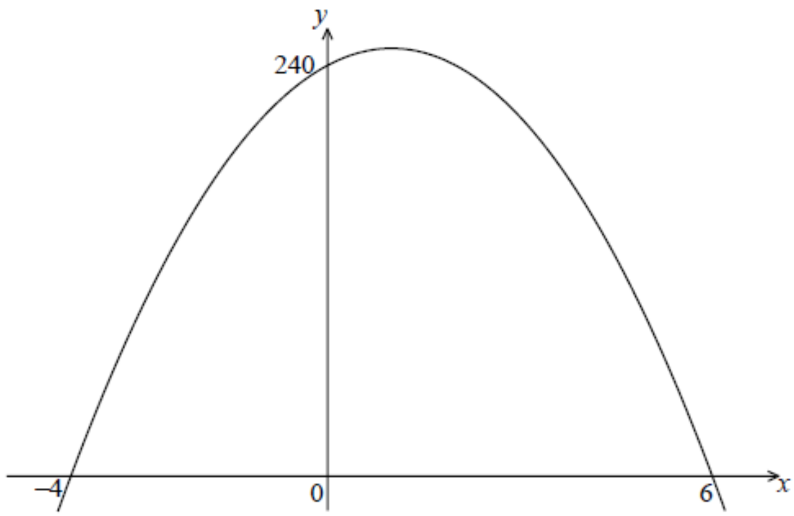
P is $(3, 0)$ ***A1A1*** ***N4***

METHOD 2

$h(x) = 2(x - 4)^2 - 2$ ***(A1)(A1)***

P is $(3, 0)$ ***A1A1*** ***N4***

5a. The following diagram shows part of the graph of a quadratic function f . *[2 marks]*



The x-intercepts are at $(-4, 0)$ and $(6, 0)$, and the y-intercept is at $(0, 240)$.

Write down $f(x)$ in the form $f(x) = -10(x - p)(x - q)$.

Markscheme

$f(x) = -10(x + 4)(x - 6)$ ***A1A1*** ***N2***

[2 marks]

5b. Find another expression for $f(x)$ in the form $f(x) = -10(x - h)^2 + k$. *[4 marks]*

Markscheme

METHOD 1

attempting to find the x-coordinate of maximum point ***(M1)***

e.g. averaging the x-intercepts, sketch, $y' = 0$, axis of symmetry

attempting to find the y-coordinate of maximum point ***(M1)***

e.g. $k = -10(1 + 4)(1 - 6)$

$f(x) = -10(x - 1)^2 + 250$ ***A1A1 N4***

METHOD 2

attempt to expand $f(x)$ ***(M1)***

e.g. $-10(x^2 - 2x - 24)$

attempt to complete the square ***(M1)***

e.g. $-10((x - 1)^2 - 1 - 24)$

$f(x) = -10(x - 1)^2 + 250$ ***A1A1 N4***

[4 marks]

5c. Show that $f(x)$ can also be written in the form $f(x) = 240 + 20x - 10x^2$. *[2 marks]*

Markscheme

attempt to simplify ***(M1)***

e.g. distributive property, $-10(x - 1)(x - 1) + 250$

correct simplification ***A1***

e.g. $-10(x^2 - 6x + 4x - 24)$, $-10(x^2 - 2x + 1) + 250$

$f(x) = 240 + 20x - 10x^2$ ***AG N0***

[2 marks]